Concrete Constructions of Real Equiangular Line Sets

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Abstract

We give some concrete constructions of real equiangular line sets. The emphasis here is on *building blocks* for certain angles which are then used to build up larger equiangular line sets. We concentrate on angles greater than or equal to 1/7.

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In these notes, we are constructing M equiangular lines in $\mathbb{R}^N$ at angle $1/\alpha$.

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        2. N=4, M=6.
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        4. N=7, M=8.

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    (A) **Building Blocks**

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        3. N=5, M=5 (Circulant)
        4. N=5, M=5 (Circulant)
        5. N=7, M=4.
        6. N=7, M=8.
        7. N=8, M=8.

(B) **Specific Dimensions**

1. N=14, M=28.
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3. N=15, M=36.
4. N=16, M=40.
5. N=19, M= 60.
6. N=21, M=45.
7. N=21, M= 45 (Using building blocks)

(C) **General sets**

1. $\mathbb{R}^N$ always has $M = N - 1$ vectors at angle 1/5.
2. $\mathbb{R}^N$ always has $M = N - 1$ vectors at angle 1/5 (Another example).
3. $\mathbb{R}^{3N+1}$ always has $4N$ vectors at angle 1/5.

(V) **Angle** 1/7.

(A) **Building Blocks**

1. N=7, M=7.
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1. N=3, M=6, angle $1/\sqrt{5}$.
2. N=5, M=10, angle 1/3.
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(X) $M = N + 1$ **vectors in** $\mathbb{R}^N$

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1. $M = N$ (Circulant self-adjoint unitary with two entries)
2. $M = N$ (Circulant self-adjoint unitary with two entries)
3. $M = N$ (Circulant unitary with two entries)

(XII) **Multiple Angles**

5. $M = 2N$, inner products $\pm 1/\sqrt{5}, 0$. (Made up of two circulants).

2 Introduction

A very old problem is:

Problem 2.1. *How many equiangular lines can we draw through the origin in $\mathbb{R}^N$?*

This means, if we choose a set of unit length vectors $\{f_m\}_{m=1}^M$, one on each line, then there is a constant c so that for all $1 \leq m \neq n \leq M$ we have

$$|\langle f_n, f_m \rangle| = c.$$

These inner products represent the cosine of the acute angle between the lines. The problem of constructing any number (especially, the maximal number) of equiangular lines in $\mathbf{R}^N$ is one of the most elementary and at the same time one of the most difficult problems in mathematics. After sixty years of research, the maximal number of equiangular lines in $\mathbf{R}^N$ is known only for 35 dimensions. For a slightly more general view of this topic see Benedetto and Kolesar [1]. This line of research was started in 1948 by Hanntjes [4] in the setting of elliptic geometry where he identified the maximal number of equiangular lines in $\mathbf{R}^N$ for $n = 2, 3$. Later, Van Lint and Seidel [7] classified the largest number of equiangular lines in $\mathbf{R}^N$ for dimensions $N \leq 7$ and at the same time emphasized the relations to discrete mathematics. In 1973, Lemmens and Seidel [6] made a comprehensive study of real equiangular line sets which is still today a fundamental piece of work. Gerzon [6] gave an upper bound for the maximal number of equiangular lines in $\mathbf{R}^N$:

Theorem 2.2 (Gerzon). *If we have M equiangular lines in $\mathbf{R}^N$ then*

$$M \leq \frac{N(N+1)}{2}.$$

We will see that in most cases there are many fewer lines than this bound gives. Also, P. Neumann [6] produced a fundamental result in the area:

Theorem 2.3 (P. Neumann). *If $\mathbf{R}^N$ has M equiangular lines at angle $1/\alpha$ and $M > 2N$, then α is an odd integer.*

Finally, there is a lower bound on the angle formed by equiangular line sets.

Theorem 2.4. *If $\{f_m\}_{m=1}^M$ is a set of norm one vectors in $\mathbf{R}^N$, then*

$$\max_{m \neq n} |\langle f_m, f_n \rangle| \geq \sqrt{\frac{M-N}{N(M-1)}}. \tag{2.1}$$

Moreover, we have equality if and only if $\{f_m\}_{m=1}^M$ is an equiangular tight frame and in this case the tight frame bound is $\frac{M}{N}$.

This inequality goes back to Welch [9]. Strohmer and Heath [8] and Holmes and Paulsen [5] give more direct arguments which also yields the "moreover" part. For some reason, in the literature there is a further assumption added to the "moreover" part of Theorem 2.4 that the vectors span $\mathbf{R}^N$. This assumption is not necessary. That is, equality in inequality 2.1 already implies that the vectors span the space [2].

The status of the equiangular line problem at this point is summarized in the following chart [6, 2, 3] where N is the dimension of the Hilbert space, M is the maximal number of equiangular lines and these will occur at the angle $1/\alpha$.

Table I: Maximal equiangular line sets

$N =$	2	3	4	5	6	7	...	13	14
$M =$	3	6	6	10	16	28	...	28	$28 - 30$
$\alpha =$	2	$\sqrt{5}$	3	3	3	3	...	3	5

$N =$	15	16	17	18	19	20
$M =$	36	≥ 40	≥ 48	≥ 48	$72 - 76*$	$92 - 96*$
$\alpha =$	5	5	5	5	5	5

$N =$	21	22	23	...	41	42	43
$M =$	126	176	276	...	276	≥ 276	344
$\alpha =$	5	5	5	...	5	5	7

The $*$ in the chart represents two cases which have been reported as solved in the literature but actually are still open.

For recent results on the equiangular line problem see [2, 3].

Here we will give concrete constructions for real equiangular line sets. The emphasis will be on constructing **building blocks** which can be put together to build larger and larger equiangular line sets.

3 $M = 3$ vectors in $\mathbb{R}^2$ at angle $\frac{1}{2}$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

4 $M = 6$ vectors in $\mathbb{R}^3$ at angle $\frac{1}{\sqrt{5}}$

$$\begin{bmatrix} 0 & \sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} \\ 0 & -\sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} \\ \sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} & 0 \\ -\sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} & 0 \\ \sqrt{\frac{5+\sqrt{5}}{10}} & 0 & \sqrt{\frac{5-\sqrt{5}}{10}} \\ \sqrt{\frac{5+\sqrt{5}}{10}} & 0 & -\sqrt{\frac{5-\sqrt{5}}{10}} \end{bmatrix}$$

5 Angle $1/3$

(A) Building Blocks

5.1 $M = 4$ vectors in $\mathbb{R}^3$

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

5.2 $M = 6$ vectors in $\mathbb{R}^4$ at angle $1/3$

$$\begin{bmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & & \\ \sqrt{\frac{1}{3}} & & \sqrt{\frac{2}{3}} & \\ \sqrt{\frac{1}{3}} & & -\sqrt{\frac{2}{3}} & \\ \sqrt{\frac{1}{3}} & & & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} & & & -\sqrt{\frac{2}{3}} \end{bmatrix}$$

5.3 $M = 8$ vectors in $\mathbb{R}^5$ at angle $1/3$

$$\sqrt{\frac{1}{3}}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

5.4 $M = 8$ vectors in $\mathbb{R}^7$ at angle $1/3$

$$\sqrt{\frac{1}{10}}\begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -1 & -1 & \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -1 & 1 & -1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & -1 & -1 & 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & 1 & -1 & -1 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & -1 & 1 & -1 & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -1 & -1 & 1 & \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

(B) Specific Dimensions

5.5 $M = 6$ **vectors in** $\mathbb{R}^4$ **at angle** $1/3$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3\sqrt{2}} & \sqrt{\frac{2}{3}} \\
-\frac{1}{3} & \frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3\sqrt{2}} & -\sqrt{\frac{2}{3}} \\
-\frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} & 0 \\
-\frac{1}{3} & -\sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & 0 \\
-\frac{1}{3} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & 0
\end{bmatrix}
$$

Remark 5.1. *The above example shows that an M element equiangular tight frame for $\mathbb{R}^N$ need not have the property that every N element subset is linearly independent? In the above example, we clearly have four vectors which sit in $\mathbb{R}^3$.*

5.6 $M = 8$ vectors in $\mathbb{R}^5$ at angle $1/3$

$$
\begin{bmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}}
\end{bmatrix}
$$

5.7 $M = 8$ vectors in $\mathbb{R}^5$ at angle $1/3$

$$
\sqrt{\frac{1}{3}}\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & -1 & 1 & 1 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & 1 & -1
\end{bmatrix}
$$

5.8 $M = 10$ vectors in $\mathbb{R}^5$ at angle $1/3$

$$
\begin{bmatrix}
\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
-\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
-\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\
-\sqrt{\frac{1}{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\
0 & \frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} \\
0 & \frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & -\sqrt{\frac{1}{2}} \\
0 & -\frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} \\
0 & \frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}}
\end{bmatrix}
$$

5.9 $M = 12$ vectors in $\mathbb{R}^6$ at angle $1/3$

$$
\begin{bmatrix}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 & \sqrt{\tfrac{1}{3}} & \sqrt{\tfrac{2}{3}} & & & & \\
 & \sqrt{\tfrac{1}{3}} & -\sqrt{\tfrac{2}{3}} & & & & \\
 & \sqrt{\tfrac{1}{3}} & & \sqrt{\tfrac{2}{3}} & & & \\
 & \sqrt{\tfrac{1}{3}} & & -\sqrt{\tfrac{2}{3}} & & & \\
 & \sqrt{\tfrac{1}{3}} & & & \sqrt{\tfrac{2}{3}} & & \\
 & \sqrt{\tfrac{1}{3}} & & & -\sqrt{\tfrac{2}{3}} & & \\
 & \sqrt{\tfrac{1}{3}} & & & & \sqrt{\tfrac{2}{3}} & \\
 & \sqrt{\tfrac{1}{3}} & & & & -\sqrt{\tfrac{2}{3}} & \\
 & & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\
 & & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\
 & & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\
 & & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}}
\end{bmatrix}
$$

5.10 $M = 12$ vectors in $\mathbb{R}^6$ at angle $1/3$ - another example

We use a building block:

$$
\frac{1}{\sqrt{3}}
\begin{bmatrix}
1 & 1 & 1 \\
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
$$

Using this we have (using + for $1/\sqrt{3}$ and - for $-1/\sqrt{3}$):

$$
\begin{bmatrix}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & + & + & + & & & \\
2 & - & + & + & & & \\
3 & + & - & + & & & \\
4 & + & + & - & & & \\
5 & & & + & + & + & \\
6 & & & - & + & + & \\
7 & & & + & - & + & \\
8 & & & + & + & - & \\
9 & + & & & & + & + \\
10 & - & & & & + & + \\
11 & + & & & & - & + \\
12 & + & & & & + & -
\end{bmatrix}
$$

5.11 $M = 16$ vectors in $\mathbb{R}^6$ at angle $1/3$

$$\frac{1}{\sqrt{6}} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1
\end{bmatrix}$$

Proof. The above matrix has mutual inner products $\pm 1/3$ since each row $2 \le m \le 16$ has two $-1's$. So their inner product with row 1 are all $1/3$. The other 15 rows come from putting $-1's$ in the 15 positions coming from 16 choose 2. So any two rows either have no $-1's$ in common and their inner product is $-1/3$ or they have one -1 in common and their inner product is $1/3$. $\qquad\square$

Remark 5.2. *This family is part of the $M = 28$ equiangular lines in $\mathbb{R}^7$ at angle $1/3$.*

5.12 $M = 16$ vectors in $\mathbb{R}^6$ at angle $1/3$ from building blocks

$$\sqrt{\frac{1}{3}} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

5.13 $M = 28$ vectors in $\mathbb{R}^7$ at angle $1/3$

.

$$\sqrt{\tfrac{1}{3}}\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}$$

5.14 $M = 28$ vectors in $\mathbb{R}^7$ at angle $1/3$ - simpler representation

Let us try this without the zeroes to see if it is clearer. Also, we add the dimension numbers.

$$\sqrt{\frac{1}{3}}$$

1	2	3	4	5	6	7
1	1	1				
-1	1	1				
1	-1	1				
1	1	-1				
		1	1	1		
		-1	1	1		
		1	-1	1		
		1	1	-1		
1				1	1	
-1				1	1	
1				-1	1	
1				1	-1	
	1		1		1	
	-1		1		1	
	1		-1		1	
	1		1		-1	
		1			1	1
		-1			1	1
		1			-1	1
		1			1	-1
1			1			1
-1			1			1
1			-1			1
1			1			-1
	1			1		1
	-1			1		1
	1			-1		1
	1			1		-1

5.15 $M = 28$ vectors in $\mathbb{R}^7$ at angle $1/3$ - Different format

Yet another way to simplify this:

First, we use our *building block* for 4 vectors in $\mathbb{R}^3$ at angle $1/3$ where $+$ means 1 and $-$ means -1.

$$BB1 = \sqrt{\frac{1}{3}} \begin{bmatrix} + & + & + \\ - & + & + \\ + & - & + \\ + & + & - \end{bmatrix}$$

Now, we put these groups of four vectors into our chart by just putting a *dot* where the columns go.

$$
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{l}
BB1\ 4 \\
BB1\ 4 \\
BB1\ 4 \\
BB1\ 4 \\
BB1\ 4 \\
BB1\ 4 \\
BB1\ 4
\end{array}
\begin{bmatrix}
\bullet & \bullet & \bullet &   &   &   &   \\
        &   & \bullet & \bullet & \bullet &   &   \\
\bullet &   &   &   & \bullet & \bullet &   \\
        & \bullet &   & \bullet &   & \bullet &   \\
        &   & \bullet &   &   & \bullet & \bullet \\
\bullet &   &   & \bullet &   &   & \bullet \\
        & \bullet &   &   & \bullet &   & \bullet
\end{bmatrix}
\end{array}
$$

5.16 $M = 28$ vectors in $\mathbb{R}^7$ at angle $1/3$ - second example

We make a bulding block by embedding our 12 vectors in $\mathbb{R}^6$ into $\mathbb{R}^7$.

$$BB9 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 & -1 & -1 \end{bmatrix}$$

Under this building block we put our standard 12 vectors in $\mathbb{R}^7$ at angle $1/3$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & & & \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & & & & & \\ \sqrt{\frac{1}{3}} & & \sqrt{\frac{2}{3}} & & & & \\ \sqrt{\frac{1}{3}} & & -\sqrt{\frac{2}{3}} & & & & \\ \sqrt{\frac{1}{3}} & & & \sqrt{\frac{2}{3}} & & & \\ \sqrt{\frac{1}{3}} & & & -\sqrt{\frac{2}{3}} & & & \\ \sqrt{\frac{1}{3}} & & & & \sqrt{\frac{2}{3}} & & \\ \sqrt{\frac{1}{3}} & & & & -\sqrt{\frac{2}{3}} & & \\ \sqrt{\frac{1}{3}} & & & & & \sqrt{\frac{2}{3}} & \\ \sqrt{\frac{1}{3}} & & & & & -\sqrt{\frac{2}{3}} & \\ \sqrt{\frac{1}{3}} & & & & & & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} & & & & & & -\sqrt{\frac{2}{3}} \end{bmatrix}$$

(C) General Sets

6 $M = 2(N-1)$ vectors in $\mathbb{R}^N$ at angle $1/3$

Proposition 6.1. *For every N, $\mathbb{R}^N$ has $2(N-1)$ equiangular lines spanning $\mathbb{R}^N$ at angle $1/3$. This family is never a tight frame.*

Proof. We will just write down the vectors:

$$
\begin{bmatrix}
\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & \cdots & 0 & 0 \\
\sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & 0 & 0 & \cdots & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & \cdots & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{2}{3}} \\
\sqrt{\frac{1}{3}} & 0 & 0 & 0 & \cdots & 0 & -\sqrt{\frac{2}{3}} \\
\end{bmatrix}
$$

$\square$

19

7 Angle $1/5$

(A) Building Blocks

7.1 $M = 4$ vectors in $\mathbb{R}^4$ at angle $1/5$

$$\begin{bmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \end{bmatrix}$$

7.2 $M = 5$ vectors in $\mathbb{R}^5$ at angle $1/5$

$$\sqrt{\frac{1}{5}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Remark 7.1. *Note that the 4×4 submatrix in the upper left corner is an orthogonal matrix.*

7.3 $M = 5$ vectors in $\mathbb{R}^5$ at angle $1/5$ and is circulant

$$\frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

7.4 $M = 5$ vectors in $\mathbb{R}^5$ at angle $1/5$ and is circulant

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}$$

7.5 $M = 4$ vectors in $\mathbb{R}^7$ at angle $1/5$

$$\begin{bmatrix}
\overset{1}{\sqrt{\tfrac{4}{10}}} & \overset{2}{\sqrt{\tfrac{1}{10}}} & \overset{3}{\sqrt{\tfrac{1}{10}}} & \overset{4}{\sqrt{\tfrac{1}{10}}} & \overset{5}{\sqrt{\tfrac{1}{10}}} & \overset{6}{\sqrt{\tfrac{1}{10}}} & \overset{7}{\sqrt{\tfrac{1}{10}}} \\
\sqrt{\tfrac{4}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{1}{10}} \\
\sqrt{\tfrac{4}{10}} & \sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} \\
\sqrt{\tfrac{4}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{1}{10}} & \sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}} & -\sqrt{\tfrac{1}{10}}
\end{bmatrix}$$

7.6 $M = 8$ vectors in R^7 at angle $1/5$

$$\sqrt{\tfrac{1}{10}}\begin{bmatrix}
1 & 1 & 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & 1 & -1 & -1 & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\
1 & -1 & 1 & -1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & -1 & -1 & 1 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\
1 & 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & 1 & -1 & -1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
1 & -1 & 1 & -1 & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & -1 & -1 & 1 & \sqrt{2} & \sqrt{2} & -\sqrt{2}
\end{bmatrix}$$

7.7 $M = 8$ vectors in $\mathbb{R}^8$ at angle $1/5$

$$\sqrt{\tfrac{1}{10}}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & \sqrt{3} \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & \sqrt{3} \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & \sqrt{3} \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & \sqrt{3} \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -\sqrt{3} \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -\sqrt{3} \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -\sqrt{3} \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & -\sqrt{3}
\end{bmatrix}$$

(B) Specific Dimensions

7.8 $M = 28$ vectors in $\mathbb{R}^{14}$ at angle $1/5$

This example is built up from building blocks. The first is:

$$BB2 = \begin{bmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \end{bmatrix}$$

This matrix will be spread out and the columns represented by "bullets".
The second is:

$$BB3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sqrt{\frac{4}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} \end{bmatrix}$$

For this matrix we will just put the first row in.
Now we use our dot trick to piece these together.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$BB3\backslash 4$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$							
$BB2\backslash 4$		$\sqrt{\frac{2}{5}}$						•	•	•				
$BB2\backslash 4$			$\sqrt{\frac{2}{5}}$							•	•	•		
$BB2\backslash 4$				$\sqrt{\frac{2}{5}}$				•				•	•	
$BB2\backslash 4$					$\sqrt{\frac{2}{5}}$				•		•		•	
$BB2\backslash 4$						$\sqrt{\frac{2}{5}}$				•		•		•
$BB2\backslash 4$							$\sqrt{\frac{2}{5}}$	•			•			•

7.9 $M = 30$ in $\mathbb{R}^{15}$ at angle $1/5$

We use a Building Block and put "bullets" each place it occurs.

$$BB4 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$BB4\backslash4$	•	•	•	•	•										
$BB4\backslash4$					•	•	•	•	•						
$BB4\backslash4$	•							•	•	•	•				
$BB4\backslash4$		•				•				•			•	•	
$BB4\backslash4$			•				•				•		•		•
$BB4\backslash4$				•				•				•		•	•

7.10 $M = 36$ vectors in $\mathbb{R}^{15}$ at angle $1/5$

First we reverse one of our Building Blocks:

$$BB5 = \begin{bmatrix} \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{2}{5}} \\ \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{2}{5}} \end{bmatrix}$$

$$BB6 = \sqrt{\frac{1}{10}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \sqrt{3} \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & \sqrt{3} \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & \sqrt{3} \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & \sqrt{3} \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -\sqrt{3} \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -\sqrt{3} \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -\sqrt{3} \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -\sqrt{3} \end{bmatrix}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
BB5 4	•	•	•					$\sqrt{\frac{2}{5}}$							
BB5 4			•	•	•			$\sqrt{\frac{2}{5}}$							
BB5 4	•			•	•				$\sqrt{\frac{2}{5}}$						
BB5 4		•		•		•				$\sqrt{\frac{2}{5}}$					
BB5 4	•		•			•					$\sqrt{\frac{2}{3}}$				
BB5 4		•		•			•					$\sqrt{\frac{2}{5}}$			
BB5 4			•		•	•							$\sqrt{\frac{2}{5}}$		
BB6 8								$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{3}{10}}$

7.11 $M = 40$ vectors in $\mathbb{R}^{16}$ at angle $1/5$

For this we need a new building block.

$$BB9 = \begin{bmatrix} \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{2}{10}} \\ \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{2}{10}} \\ -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{2}{10}} \\ -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{2}{10}} \end{bmatrix}$$

In the next chart, the first seven rows are building block 5 and contain four vectors each. Row 8 is Building block 6 and has eight vectors. And Row 9 is Building block 9 and has four vectors.

$$\begin{bmatrix}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
 & \bullet & \bullet & \bullet & & & & & \sqrt{\frac{2}{5}} & & & & & & & & \\
 & & & \bullet & \bullet & \bullet & & & & \sqrt{\frac{2}{5}} & & & & & & & \\
 & \bullet & & & \bullet & \bullet & & & & & \sqrt{\frac{2}{5}} & & & & & & \\
 & & \bullet & & & \bullet & & \bullet & & & & \sqrt{\frac{2}{5}} & & & & & \\
 & \bullet & & & \bullet & & & \bullet & & & & & \sqrt{\frac{2}{3}} & & & & \\
 & \bullet & & & \bullet & & & \bullet & & & & & & \sqrt{\frac{2}{5}} & & & \\
 & & \bullet & & & \bullet & \bullet & & & & & & & & \sqrt{\frac{2}{5}} & & \\
 & & & & & & & & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{3}{10}} & \\
 & \sqrt{\frac{1}{5}} & & & & & & & \pm\sqrt{\frac{1}{10}} & & \pm\sqrt{\frac{1}{10}} & & \pm\sqrt{\frac{1}{10}} & \pm\sqrt{\frac{1}{10}} & & \pm\sqrt{\frac{2}{5}}
\end{bmatrix}$$

On the next page we will give a hint about how to prove this is an equiangular line set.

The first 36 vectors represent 36 equiangular lines in $\mathbb{R}^{15}$ and this is easily checked by sight. Also, the last four vectors have $\sqrt{\frac{1}{5}}$ in the first position and this hits exactly the groups in rows 1,3 and 5 in eactly one position - the first position and this yields $1/5$. The last group hits the elements in rows 2,4,6, and 7 in exactly one place and produces

$$\sqrt{\frac{2}{5}} \cdot \sqrt{\frac{1}{10}} = \frac{1}{5}.$$

So we only need to check how the last four vectors interact with the eight just before them. But these vectors all have exactly four coordinates in common (9,11,13, and 14) and they are respectively:

$$\begin{bmatrix} + & + & + & + \\ + & - & - & + \\ - & - & - & - \\ - & + & + & - \\ + & + & - & - \\ + & - & + & - \\ - & - & + & + \\ - & + & - & + \end{bmatrix}$$

and

$$\begin{bmatrix} - & - & - & + \\ + & + & + & - \\ - & + & - & - \\ + & - & + & + \end{bmatrix}$$

Now, a visual check shows that the inner products of the rows in the second group with the rows in the first group will yield

$$\pm \left[\sqrt{\frac{1}{10}} \cdot \sqrt{\frac{1}{10}} + \sqrt{\frac{1}{10}} \cdot \sqrt{\frac{1}{10}} \right] = \pm\frac{1}{5}.$$

7.12 $M = 60$ vectors in $\mathbb{R}^{19}$ at angle $1/5$

We use a Building Block and put "bullets" each place it occurs.

$$
BB4 = \frac{1}{\sqrt{5}}
\begin{bmatrix}
-1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & & & \\
\bullet & & & & & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & \\
\bullet & & & & & & & & & \bullet & \bullet & \bullet & \bullet & & & & & & \\
\bullet & & & & & & & & & & & & & \bullet & \bullet & \bullet & \bullet & & \\
& \bullet & & & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & \\
& & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & & & \bullet & \\
& & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & & \bullet & \\
& \bullet & & & & & & \bullet & & & & \bullet & & & & \bullet & \bullet & & \\
& \bullet & & & & & \bullet & & & & & \bullet & & & & & \bullet & & \bullet \\
& & \bullet & & & & & \bullet & & & & & \bullet & \bullet & & & & & \bullet \\
& & & \bullet & & \bullet & \bullet & & & & & \bullet & & & \bullet & & & & \bullet \\
& & & & \bullet & \bullet & & & & \bullet & & & & & & \bullet & & & \bullet
\end{bmatrix}
$$
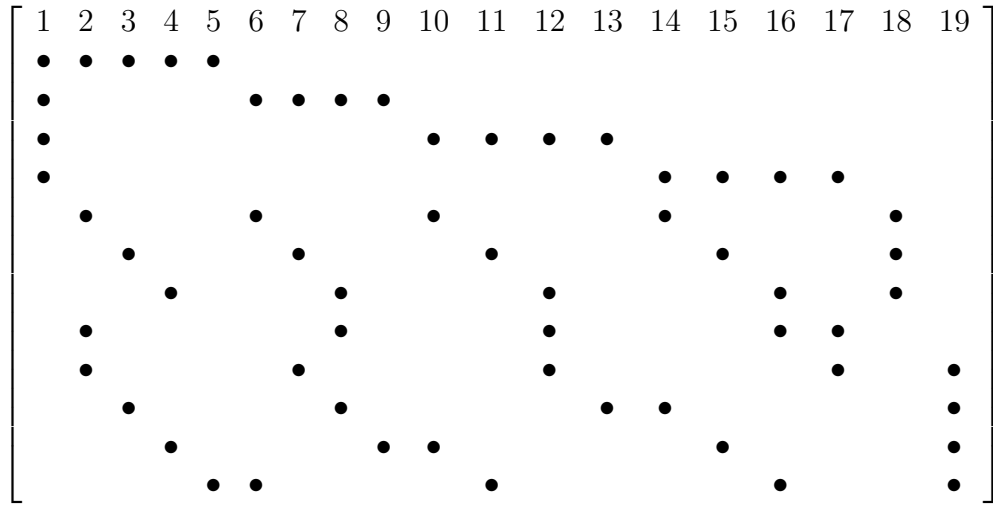
7.13 $M = 45$ vectors in $\mathbb{R}^{21}$ at angle $1/5$

We use two building blocks. The first is a 4×4 orthogonal matrix with an extra column - and it is represented in our matrix by "bullets".

$$BB7 = \sqrt{\frac{1}{5}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The second building block is a 5×5 matrix - and it is represented by "*".

$$BB8 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & & & & & \\
\bullet & & & & & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & \\
\bullet & & & & & & & & & \bullet & \bullet & \bullet & \bullet & & & & & & & & \\
\bullet & & & & & & & & & & & & & \bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & & & & & & & & & & & & & & & & & \bullet & \bullet & \bullet & \bullet \\
& * & & & & * & & & & * & & & & * & & & & * & & & \\
& * & & & & & * & & & & * & & & & * & & & & * & & \\
& * & & & & & & * & & & & * & & & & * & & & & * & \\
& * & & & & & & & * & & & & * & & & & * & & & & *
\end{bmatrix}$$

7.14 $M = 45$ vectors in $\mathbb{R}^{21}$ at angle $1/5$

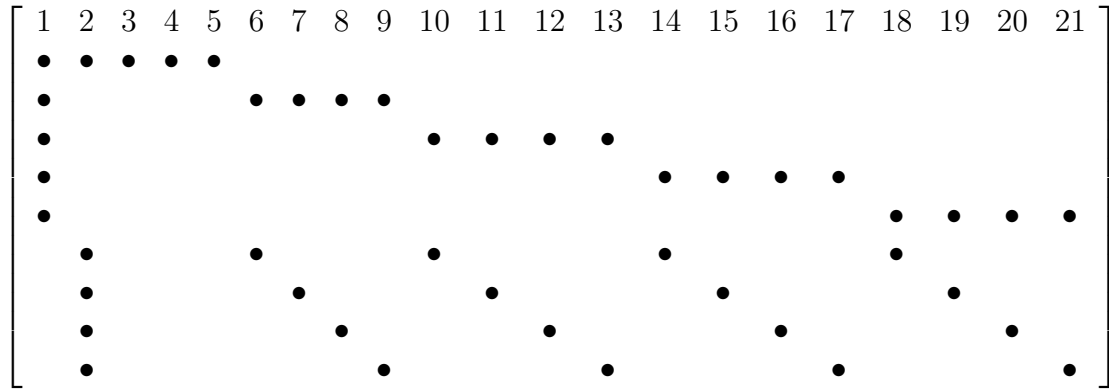We use the a Building Block and put "bullets" each place it appears.

$$BB4 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & & & & & \\ \bullet & & & & & \bullet & \bullet & \bullet & \bullet & & & & & & & & & & & & \\ \bullet & & & & & & & & & \bullet & \bullet & \bullet & \bullet & & & & & & & & \\ \bullet & & & & & & & & & & & & & \bullet & \bullet & \bullet & \bullet & & & & \\ \bullet & & & & & & & & & & & & & & & & & \bullet & \bullet & \bullet & \bullet \\ & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & & & \\ & \bullet & & & & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & & \\ & \bullet & & & & & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet & \\ & \bullet & & & & & & & \bullet & & & & \bullet & & & & \bullet & & & & \bullet \end{bmatrix}$$

(C) General Sets

7.15 $N-1$ vectors in $\mathbb{R}^N$ at angle $1/5$

$$
\begin{bmatrix}
1 & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{2}{5}} & \sqrt{\tfrac{2}{5}} & & & & \cdots \\
2 & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{2}{5}} & -\sqrt{\tfrac{2}{5}} & & & & \cdots \\
3 & \sqrt{\tfrac{1}{5}} & & \sqrt{\tfrac{2}{5}} & \sqrt{\tfrac{2}{5}} & & & \cdots \\
4 & \sqrt{\tfrac{1}{5}} & & \sqrt{\tfrac{2}{5}} & -\sqrt{\tfrac{2}{5}} & & & \cdots \\
5 & \sqrt{\tfrac{1}{5}} & & & \sqrt{\tfrac{2}{5}} & \sqrt{\tfrac{2}{5}} & & \cdots \\
6 & \sqrt{\tfrac{1}{5}} & & & \sqrt{\tfrac{2}{5}} & -\sqrt{\tfrac{2}{5}} & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

7.16 $N-1$ vectors in $\mathbb{R}^N$ at angle $1/5$

$$
\begin{bmatrix}
1 & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{3}{5}} & & & & \cdots \\
2 & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{1}{5}} & -\sqrt{\tfrac{3}{5}} & & & & \cdots \\
3 & \sqrt{\tfrac{1}{5}} & & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{3}{5}} & & & \cdots \\
4 & \sqrt{\tfrac{1}{5}} & & \sqrt{\tfrac{1}{5}} & -\sqrt{\tfrac{3}{5}} & & & \cdots \\
5 & \sqrt{\tfrac{1}{5}} & & & \sqrt{\tfrac{1}{5}} & \sqrt{\tfrac{3}{5}} & & \cdots \\
6 & \sqrt{\tfrac{1}{5}} & & & \sqrt{\tfrac{1}{5}} & -\sqrt{\tfrac{3}{5}} & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

7.17 $M = 4N$ vectors in $\mathbb{R}^{3N+1}$ at angle $1/5$

We use our building block

$$BB5 = \begin{bmatrix} \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{2}{5}} \\ \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{2}{5}} \end{bmatrix}$$

We use "bullets" to indicate where these columns go.

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & & & & & & & \cdots \\ \bullet & & & & \bullet & \bullet & \bullet & & & & \cdots \\ \bullet & & & & & & & \bullet & \bullet & \bullet & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

8 Angle $1/7$

(A) Building Blocks

8.1 $M = 7$ vectors in $\mathbb{R}^7$ at angle $1/7$

$$\frac{1}{\sqrt{7}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

8.2 $M = 8$ vectors in $\mathbb{R}^7$ at angle $1/7$

$$\sqrt{\frac{1}{10}}\begin{bmatrix} 1 & 1 & 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -1 & -1 & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & -1 & 1 & -1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & -1 & -1 & 1 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & 1 & -1 & -1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -1 & 1 & -1 & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -1 & -1 & 1 & \sqrt{2} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

9 $M = 2N$ vectors in $\mathbb{R}^N$

9.1 $M = 6$ vectors in $\mathbb{R}^3$ at angle $1/\sqrt{5}$

$$
\begin{bmatrix}
0 & \sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} \\
0 & -\sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} \\
\sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} & 0 \\
-\sqrt{\frac{5-\sqrt{5}}{10}} & \sqrt{\frac{5+\sqrt{5}}{10}} & 0 \\
\sqrt{\frac{5+\sqrt{5}}{10}} & 0 & \sqrt{\frac{5-\sqrt{5}}{10}} \\
\sqrt{\frac{5+\sqrt{5}}{10}} & 0 & -\sqrt{\frac{5-\sqrt{5}}{10}}
\end{bmatrix}
$$

9.2 $M = 10$ vectors in $\mathbb{R}^5$ at angle $1/3$

$$
\begin{bmatrix}
\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
-\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
-\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
\sqrt{\frac{1}{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\
-\sqrt{\frac{1}{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\
0 & \frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} \\
0 & \frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & -\sqrt{\frac{1}{2}} \\
0 & -\frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} \\
0 & \frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & -\frac{1}{3}\sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}}
\end{bmatrix}
$$

9.3 $M = 12$ **vectors in** $\mathbb{R}^6$ **at angle** $1/3$

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\sqrt{\tfrac{1}{3}} & \sqrt{\tfrac{2}{3}} & & & & \\
\sqrt{\tfrac{1}{3}} & -\sqrt{\tfrac{2}{3}} & & & & \\
\sqrt{\tfrac{1}{3}} & & \sqrt{\tfrac{2}{3}} & & & \\
\sqrt{\tfrac{1}{3}} & & -\sqrt{\tfrac{2}{3}} & & & \\
\sqrt{\tfrac{1}{3}} & & & \sqrt{\tfrac{2}{3}} & & \\
\sqrt{\tfrac{1}{3}} & & & -\sqrt{\tfrac{2}{3}} & & \\
\sqrt{\tfrac{1}{3}} & & & & \sqrt{\tfrac{2}{3}} & \\
\sqrt{\tfrac{1}{3}} & & & & -\sqrt{\tfrac{2}{3}} & \\
 & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\
 & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\
 & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\
 & \tfrac{1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{-1}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}}
\end{bmatrix}
$$

9.4 $M = 28$ **vectors in** $\mathbb{R}^{14}$ **at angle** $1/3$

First, we use our *building block* for 4 vectors in $\mathbb{R}^3$ at angle $1/3$ where $+$ means 1 and $-$
means -1.

$$
BB1 = \sqrt{\frac{1}{3}}
\begin{bmatrix}
+ & + & + \\
- & + & + \\
+ & - & + \\
+ & + & -
\end{bmatrix}
$$

Now, we put these groups of four vectors into our chart by just putting a *dot* where the
columns go.

$$
\begin{array}{c}
 \quad 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7 \\
\begin{array}{c}
BB1\,4 \\
BB1\,4 \\
BB1\,4 \\
BB1\,4 \\
BB1\,4 \\
BB1\,4 \\
BB1\,4
\end{array}
\begin{bmatrix}
\bullet & \bullet & \bullet & & & & \\
 & & \bullet & \bullet & \bullet & & \\
\bullet & & & & \bullet & \bullet & \\
 & \bullet & & \bullet & & & \bullet \\
 & & \bullet & & & \bullet & \bullet \\
\bullet & & & \bullet & & & \bullet \\
 & \bullet & & & \bullet & & \bullet
\end{bmatrix}
\end{array}
$$

9.5 $M = 28$ vectors in $\mathbb{R}^{14}$ at angle $1/5$

This example is built up from building blocks. The first is:

$$BB2 = \begin{bmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} & -\sqrt{\frac{1}{5}} \end{bmatrix}$$

This matrix will be spread out and the columns represented by "bullets".
The second is:

$$BB3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sqrt{\frac{4}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{10}} \end{bmatrix}$$

For this matrix we will just put the first row in.
Now we use our dot trick to piece these together.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$BB3\backslash 4$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$							
$BB2\backslash 4$		$\sqrt{\frac{2}{5}}$						$\bullet$	$\bullet$	$\bullet$				
$BB2\backslash 4$			$\sqrt{\frac{2}{5}}$							$\bullet$	$\bullet$	$\bullet$		
$BB2\backslash 4$				$\sqrt{\frac{2}{5}}$				$\bullet$				$\bullet$		
$BB2\backslash 4$					$\sqrt{\frac{2}{5}}$					$\bullet$		$\bullet$		$\bullet$
$BB2\backslash 4$						$\sqrt{\frac{2}{5}}$					$\bullet$		$\bullet$	$\bullet$
$BB2\backslash 4$							$\sqrt{\frac{2}{5}}$	$\bullet$			$\bullet$			$\bullet$

10 $M = N + 1$ vectors in $\mathbb{R}^N$

Proposition 10.1. *For every natural number N, the $(N{+}1){\times}(N{+}1)$ matrix below represents $N + 1$ unit norm vectors $\{f_m\}_{m=1}^{N+1}$ in $\mathbb{R}^{N+1}$ giving an equiangular tight frame for an N-dimensional space and satisfying:*

(1) $\|f_m\| = 1$.

(2) For any $n \neq m$, $\langle f_m, f_n \rangle = \frac{-1}{N}$.

Proof. For any m we have

$$\|f_m\|^2 = \frac{1}{N(N+1)}[N^2 + N] = 1.$$

Also,

$$\langle f_m, f_n \rangle = \frac{-2N + N - 1}{N(N+1)} = \frac{-(N+1)}{N(N+1)} = \frac{-1}{N}.$$

$\square$

11 Unitary Matrices

11.1 Circulant self-adjoint Matrix

$$\begin{bmatrix} -b & a & a & \cdots & a \\ a & -b & a & \cdots & a \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a & a & a & \cdots & a \\ a & a & a & \cdots & -b \end{bmatrix}$$

11.2 Circulant self-adjoint Unitary Matrix

If we rotate the rows of the matrix we can get matrices for every $2N$-dimensional Hilbert space yielding a circulant, self-adjoint unitary matrix.

$$\left[\begin{array}{cccc|cccc} a & a & \ldots & a & -b & a & \ldots & a \\ a & a & \ldots & a & a & -b & \ldots & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & a & \ldots & a & a & a & \ldots & -b \\ \hline -b & a & \ldots & a & a & a & \ldots & a \\ a & -b & \ldots & a & a & a & \ldots & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & a & \ldots & -b & a & a & \ldots & a \end{array}\right]$$

Proof. We have for any two rows $n \neq m$ of this matrix:

$$\langle f_n, f_m \rangle = (2N - 2)a2 - 2ab = a[(2(N - 1)a - 2b].$$

So this is zero if $b = (N - 1)a$. $\qquad\square$

11.3 Another example of circulant matrices

$$\begin{bmatrix} a & -b & a & \cdots & a \\ a & a & -b & \cdots & a \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a & a & a & \cdots & -b \\ -b & a & a & \cdots & a \end{bmatrix}$$

12 Multiple angles

12.1 $M = 2N$ vectors at two angles: $1/\sqrt{5}, 0$

Proposition 12.1. *For every N, $\mathbb{R}^N$ has $2N$ lines at three angles: $\pm 1/\sqrt{5}, 0$. These lines span $\mathbb{R}^N$.*

Proof. We let

$$x = \sqrt{\frac{5 - \sqrt{5}}{10}} \quad y = \sqrt{\frac{5 + \sqrt{5}}{10}}.$$

We will just write down the vectors:

$$
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
\vdots \\
f_{2N-3} \\
f_{2N-2} \\
f_{2N-1} \\
f_{2N}.
\end{bmatrix}
=
\begin{bmatrix}
x & y & 0 & 0 & \cdots & 0 & 0 & 0 \\
-x & y & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & x & y & 0 & \cdots & 0 & 0 & 0 \\
0 & -x & y & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & x & y & \cdots & 0 & 0 & 0 \\
0 & 0 & -x & y & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & x & y \\
0 & 0 & 0 & 0 & \cdots & 0 & -x & y \\
y & 0 & 0 & 0 & \cdots & 0 & 0 & x \\
y & 0 & 0 & 0 & \cdots & 0 & 0 & -x
\end{bmatrix}
$$

It is obvious that these lines span $\mathbb{R}^N$. $\qquad\square$

Corollary 12.2. *For every N, the $2N$ unit vectors at angles $1/\sqrt{5}, 0$ can be divided into two sets of circulant vectors.*

Proof. The two sets are:

$$
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_N
\end{bmatrix}
=
\begin{bmatrix}
x & y & 0 & 0 & \cdots & 0 & 0 \\
0 & x & y & 0 & \cdots & 0 & 0 \\
0 & 0 & x & y & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & x & y \\
y & 0 & 0 & 0 & \cdots & 0 & x
\end{bmatrix}
$$

$$
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_N
\end{bmatrix}
=
\begin{bmatrix}
-x & y & 0 & 0 & \cdots & 0 & 0 \\
0 & -x & y & 0 & \cdots & 0 & 0 \\
0 & 0 & -x & y & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -x & y \\
y & 0 & 0 & 0 & \cdots & 0 & -x
\end{bmatrix}
$$

$\qquad\square$

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